§4.2 4-manifold invariants from 5-branes

Choosing 6d-spacetime to be $M_4 \times \Sigma$ gives:

$$SO(6)_E \longrightarrow \underset{\underset{SU(2)_\ell \times SU(2)_r}{\wr\!\wr}}{SO(4)_E} \times SO(2)_\Sigma$$

R-symmetry group:

	$SO(5)_R$
B-field	1
scalars	5
Weyl ferm.	4

We have following branching rules:

$$SO(6)_E \longrightarrow SU(2)_\ell \times SU(2)_r \times U(1)_\Sigma$$

$$4_+ \longrightarrow (2,1)^{+1} \oplus (1,2)^{-1}$$

$$4_- \longrightarrow (2,1)^{-1} \oplus (1,2)^{+1}$$

$$6 \longrightarrow (2,2)^0 \oplus (1,1)^{+2} \oplus (1,1)^{-2}$$

and

$$SO(5)_R \longrightarrow SU(2)_R \times U(1)_t$$

$$5 \longrightarrow 3^0 \oplus 1^{\pm 2}$$

$$4 \longrightarrow 2^{+1} \oplus 2^{-1}$$

In order to topologically twist the theory, embed $M_4 \times \Sigma$ into

$$\underbrace{\Lambda_+^2 M_4}_{G_2\text{-hol.}} \times \underbrace{T^*\Sigma}_{\text{loc. K3}}$$

Decompose the R-sym. as

$$SO(5)_R \longrightarrow SO(3)_R \times SU(2)_f$$

then the fermions of 6d (2,0) trf. as:

$$SO(6)_E \times SO(5)_R \longrightarrow SU(2)_\ell \times SU(2)_r \times SU(2)_R \times U(1)_\Sigma \times U(1)_f$$

fermions $(4_+, 4)$ $\qquad (2,1,2)^{(1,\pm 1)} \oplus (1,2,2)^{(-1,\pm 1)}$

Note when $M_4 = \mathbb{R} \times M_3$ or $M_4 = S^1 \times M_3$, the rotation symmetry on M_3 is a diagonal subgroup $SU(2)_M \subset SU(2)_\ell \times SU(2)_r$

$\longrightarrow$ replace $SU(2)_r$ with diagonal subgroup $SU(2)_r' \subset SU(2)_r \times SU(2)_R$

$\longrightarrow$ new transformation rules:

$$SO(6)_E \times SO(5)_R \longrightarrow SU(2)_\ell \times SU(2)_r' \times U(1)_\Sigma \times U(1)_f$$

fermions: $(4_+, 4) \longrightarrow (2,2)^{(1,\pm 1)} \oplus (1,3)^{(-1,\pm 1)} \oplus (1,1)^{(-1,\pm 1)}$

$\to$ two preserved supercharges are chiral $\to$ 2d $\mathcal{N}=(0,2)$ SUSY along Σ

If $M_4 = \mathbb{R}\times M_3$, then before the twist we have

$$SO(6)_E \times SO(5)_R \longrightarrow SU(2)_M \times SU(2)_R \times U(1)_\Sigma \times U(1)_t$$
$$(4_+, 4) \longrightarrow (2,2)^{(\pm 1, \pm 1)}$$

Instead of twisting along M_4 (or M_3) we can start with a partial top. twist along Σ

$\longrightarrow$ $U(1)_\Sigma$ is replaced with diag subgroup $U(1)'_\Sigma \subset U(1)_\Sigma \times U(1)_t$

Then

$$SO(6)_E \times SO(5)_R \to SU(2)_\ell \times SU(2)_r \times SU(2)_R \times U(1)'_\Sigma \times U(1)_t$$
$$(4_+, 4) \to (2,1,2)^{(2,1)} \oplus \underline{(2,1,2)^{(0,-1)}} \oplus \underline{(1,2,2)^{(0,1)}}$$
$$\oplus\, (1,2,2)^{(-2,-1)}$$

$U(1)'_\Sigma$ - singlets

transform as supercharges of 4d $\mathcal{N}=2$ th on M_4 with R-sym $SU(2)_R \times U(1)_t$

Replacing $SU(2)_r$ with the diagonal subgroup $SU(2)_r' \subset SU(2)_r \times SU(2)_R$, we get

$$SO(6)_E \times SO(5)_R \longrightarrow SU(2)_\ell \times SU(2)_r' \times U(1)_\Sigma' \times U(1)_t$$

$$(4_t, 4) \longrightarrow (2,2)^{(2,1)} \oplus (2,2)^{(0,-1)} \oplus (1,3)^{(0,1)} \oplus \underline{(1,1)^{(0,1)}} \oplus (1,3)^{(-2,-1)} \oplus (1,1)^{(-2,-1)}$$

only one supercharge is singlet under symmetries of M_4 <u>and</u> Σ, denote by $\mathcal{Q}$

Denoting generators of $U(1)_\Sigma'$ and $U(1)_t$ by P and R_t, respectively, we can read off $\mathcal{Q}^2 = 0$, $[R_t, \mathcal{Q}] = \mathcal{Q}$, $[P, \mathcal{Q}] = 0$

When $M_4 = \mathbb{R} \times M_3$ or $M_4 = S^1 \times M_3$, we have two scalar supercharges; the second one arises from the decomposition $2 \otimes 2 = 3 \oplus 1$ with respect to $SU(2)_M$ or, equivalently, from twist along Σ

VW partition function as a CS wave-function

$T_{U(1)}[M_3]$ for plumbed $M_3 \to$ quiver CS-th

Alternatively, can think of M_4 with $\partial M_4 = M_3$ and intersection form on M_4 given by Q.

Consider quantization of abelian CS-th on $T^2 \times \mathbb{R} \to$ There are $|\mathrm{Coker}\, Q| = |H|$ states on the torus and they correspond to basic Wilson lines

$$\prod_{i \text{ vertices}} x_i^{h_i} \in \mathbb{C}[x_1, \dots, x_{b_2}] \Big/ \Big\{ \prod_j x_j^{Q^{ji}} - 1 \Big\},$$

$$h \in H$$

inserted in the solid torus bounded by T^2

One can also specify a wave-function of such states

$\longrightarrow$ let $|x\rangle \in \mathcal{H}_{T[M_3]}(T^2)$ be a state with given holonomies and $|h\rangle \in \mathcal{H}_{T[M_3]}(T^2)$ a state created by a Wilson line.

$$\longrightarrow \quad \langle h|x\rangle = \Psi_h(x) = \sum_{\lambda \in \Lambda + h + \omega_2/2} q^{-\frac{1}{2}(\lambda,\lambda) - b_2/8}\, x^\lambda \qquad (*)$$

where $q = e^{2\pi i \tau}$ and $(\cdot,\cdot)$ is bilinear form on Λ given by Q and extended to $\Lambda^* \subset \mathbb{Q} \otimes_{\mathbb{Z}} \Lambda$.

The element $\omega_2 \in \Lambda^*$ has to be chosen such that

$$\omega_2(\lambda) = (\lambda,\lambda) \mod 2, \ \forall \lambda \in \Lambda$$

$\longrightarrow$ fixes $[\omega_2] \in \Lambda^*/2\Lambda^*$

(requirement arise from quantization of abelian CS-th)

The overall factor $q^{-b_2/8}$ is chosen so that the wave function has nice properties under S- and T-transformations

In particular, the T-matrix is given by

$$T_{hh'} = \delta_{hh'}\, e^{-\pi i [(h+w_2/2,\, h+w_2/2) - b_2/4]}$$

and is an invariant of M_3.

Up to an overall factor, $(*)$ is equal to the partition function of abelian VW theory on M_4 with a boundary condition labeled by $h \in H_1(M_3)$:

$$Z_{VW}[M_4](q, x)_h \sim \sum_{[F/2\pi] \in \Lambda + h + w_2/2} q^{\frac{1}{8\pi^2}\int F\wedge F}\, x^{[F/2\pi]} \sim \mathcal{Y}_h(x)$$

On the 4-manifold side, the fugacities x_i are chemical potentials for the first Chern class of the gauge connection on M_4.

h labels choice of flat connection ρ on M_3

$$\rightarrow \quad Z_{VW}[M_4](q;x) \in \mathcal{H}_{VW}(M_3)$$

$$\cong \mathcal{H}_{T[T^2]}(M_3) = \mathcal{H}_{T[M_3]}(T^2)$$

so that

$$Z_{VW}[M_4](q;x)_h = \langle h | Z_{VW}[M_4](q;x)$$

The wave-functions have the following q-expansions:

$$\psi_h(x) = q^{-\Delta(h)/2} + \cdots$$

conformal dimensions of primaries of boundary CFT, chiral $U(1)^{b_2}$ WZW-th.

$$\Delta(h) = \max_{\lambda \in \Lambda + h + \omega_2/2} \left[(\lambda,\lambda)_Q + b_2/4 \right]$$